8 MUST KNOW QUESTIONS TO ACE FUNCTIONS

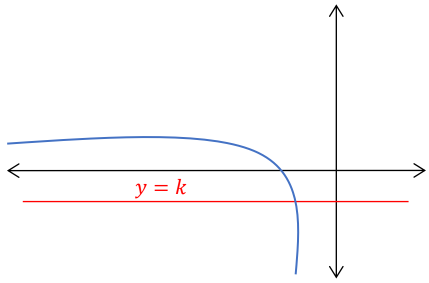
1	**[YIJC/2021/P1/Q4]** Functions f and g are defined by $f : x \mapsto 2 + \frac{3}{1-x^2}$, $\quad x \in \mathbb{R}, x < -1$ $g : x \mapsto -x^2 + 6x + a$, $\quad x \in \mathbb{R}, x \geq 0$, where a is a positive integer. **(i)** Show that f has an inverse. **(ii)** Find $f^{-1}(x)$ and state the domain of f^{-1}. **(iii)** State whether the composite function fg exists, justifying your answer.	 **[1]** **[3]** **[3]**
2	**[ACJC/2022/P1/Q6]** The function h is defined by h : $x \mapsto \lvert \ln(2x) \rvert + 1, x \in \mathbb{R}, 0 < x \leq \lambda.$, **(i)** Find the maximum value of λ for which the inverse function h exist. **(ii)** Using $\lambda = \frac{1}{2}$, **(a)** find h^{-1} and state its domain, **(b)** sketch the graph of $y = hh^{-1}(x)$, **(c)** find the solution set for $h(x) = h^{-1}(x)$.	 **[1]** **[4]** **[1]** **[1]**
3	**[N2016/P1/Q10]** **(a)** The function f is given by $f: x \mapsto 1 + \sqrt{x}$, for $x \in \mathbb{R}, x \geq 0$. **(i)** Find $f^{-1}(x)$ and state the domain of f^{-1}. **(ii)** Show that if ff$(x) = x$ then $x^3 - 4x^2 + 4x - 1 = 0$. Hence find the value of x for which ff$(x) = x$. Explain why this value of x satisfies the equation f$(x) = f^{-1}(x)$. **(b)** The function g, with domain the set of non-negative integers, is given by $g(n) = \begin{cases} 1 & \text{for } n = 0, \\ 2 + g\left(\frac{1}{2}n\right) & \text{for } n = \text{even}, \\ 1 + g(n-1) & \text{for } n = \text{odd} \end{cases}$ **(i)** Find g(4), g(7) and g(12). **(ii)** Does g have an inverse? Justify your answer.	 **[3]** **[5]** **[3]** **[2]**
4	**[NYJC/2023/Prelims/P1/Q3]** Functions f and g are defined by f : $x \mapsto \frac{x+a}{x+b}$, $\quad$ for $x \in \mathbb{R}, x \neq -b, a \neq -1$, g : $x \mapsto x$, $\quad$ for $x \in \mathbb{R}$. It is given that ff = g. Find the value of b. Find $f^{-1}(x)$ in terms of x and a.	 **[5]**

5	**[NYJC/2023/P1/Q6]** The functions f and g are defined by $\text{f} : x \mapsto \frac{3x+18}{2x-3}, \quad x \in \mathbb{R}, \ -6 \le x \le 0, \ x > \frac{3}{2},$ $\text{g} : x \mapsto ex, \quad x \in \mathbb{R}.$ **(a)** Find $f^{-1}(x)$ and state its domain. **[3]** **(b)** State the set of values of x such that $f(x) = f^{-1}(x)$, **[1]** **(c)** On the same diagram, sketch the graphs of $y = f(x)$ and $y = f^{2024}(x)$, giving the coordinates of any axial intercepts and the equations of any asymptotes. **[4]** **(d)** State the exact range of $g^{23}f$. **[1]**
6	**[EJC/2019/P2/Q4]** The ceiling function maps a real number x to the least integer greater than or equal to x. Denote the ceiling function as $\lceil x \rceil$. For example, $\lceil 2.1 \rceil = 3$ and $\lceil -3.8 \rceil = -3$. The function f is defined by $f(x) = \begin{cases} \lceil x \rceil & \text{for } x \in \mathbb{R}, \ -2 < x \le 1, \\ 0 & \text{for } x \in \mathbb{R}, \ 1 < x \le 2. \end{cases}$ **(i)** Find the value of $f(-1.4)$. **[1]** **(ii)** Sketch the graph of y = f(x) for $-2 < x \le 2$. **[2]** **(iii)** Does f^{-1} exist? Justify your answer. **[1]** **(iv)** Find the range of f. **[1]** The function g is defined as $\text{g} : x \mapsto \frac{ax-3}{x-a}, x \in \mathbb{R}, x \ne a,$ where $a > 0, a \ne 3$. **(v)** Find $g^2(x)$. Hence, or otherwise, evaluate $g^{2019}(5)$, leaving your answer in a if necessary. **[4]** **(vi)** Given that $a = 3$, find the range of gf. **[1]**
7	**[NJC/2018/P1/Q6]** **(a)** The function f is defined by $\text{f} : x \mapsto 2x^2 - \lambda x - 3$, where $x \in \mathbb{R}, \ \frac{7}{4} < x < 5,$ where λ is a real constant. Find the set of possible values of λ such that f^{-1} exists. **[2]** **(b)** The function g is defined by $\text{g}(x) = \begin{cases} (x-1)^2, & \text{for } 0 \le x \le 1, \\ 2\log_2 x & \text{for } 1 < x \le 2. \end{cases}$ **(i)** Sketch the graph of $y = g(x)$, labelling clearly the coordinates of the end-points and the points where the curve crosses the x-axis, if any. **[2]** **(ii)** Hence solve the inequality $1 < g(x) \le 2$ exactly. **[2]** **(iii)** Given that g^2 exists, define g^2 in a similar form as g. **[3]**

8	**[RVHS/2018/P1/Q4]** A function is said to be self-inverse when $f = f^{-1}$ for all x in the domain of f. Given that the function f is defined by $\mathrm{f} : x \mapsto \frac{ax+b}{cx-a}$, for $x \in \mathbb{R}, x \neq \frac{a}{c}$, where a, b and c are positive constants. **(i)** Show that f is self-inverse. **[2]** **(ii)** Using the result of part (i), deduce $f^2(x)$ and state the range of f^2. **[2]** **(iii)** Solve the equation $f^{-1}(x) = x$, leaving your answers in the exact form. **[3]** For the rest of the question, let $a = 2, b = 5$ and $c = 3$. The function g is defined by $\mathrm{g} : x \mapsto e^x + 2$ for $x \in \mathbb{R}$. **(iv)** Show that the composite function fg exists, justifying your answer clearly. **[1]** **(v)** By considering the graph of f, or otherwise, find the exact range of fg. **[2]**

Solutions

1 **(i)**



Any horizontal line $y = k$, where $k \in \mathbb{R}$, cuts the graph of f **at most once**, therefore f is one-one and f has an inverse.

(ii) Let $y = 2 + \frac{3}{1-x^2}$

$\frac{3}{1-x^2} = y - 2$

$1 - x^2 = \frac{3}{y-2}$

$x = \pm\sqrt{1 - \frac{3}{y-2}}$ or $x = -\sqrt{1 - \frac{3}{y-2}}$

(rej. since $x < -1$)

$\therefore f^{-1}(x) = -\sqrt{1 - \frac{3}{x-2}}, x < 2$

(iii) $g(x) = -x^2 + 6x + a$

$g(x) = -(x-3)^2 + a + 9$

$D_f = (-\infty, -1), R_g = (-\infty, a+9]$

Since $a + 9 > -1$ for $a > 0, R_g \not\subseteq D_f$, therefore, fg does not exist

2 **(i)** Maximum value of λ for which the inverse function h exist is $\frac{1}{2}$

(ii) **(a)** When $0 < x \leq \frac{1}{2}$, $\ln(2x) < 0$

$\Rightarrow |ln(2x)| = -ln(2x)$

$y = -ln(2x) + 1$

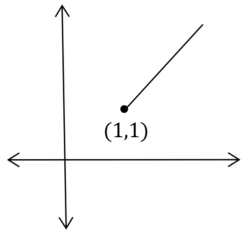
$ln(2x) = 1 - y$

$2x = e^{1-y}$

$x = \frac{1}{2}e^{1-y}$

$h^{-1}(x) = \frac{1}{2}e^{1-x}, x \geq 1$

(b)



(c) **<u>Method 1</u>**

$R_h = [1, \infty)$ and $R_{h^{-1}} = (0, \frac{1}{2}]$

$R_h \cap R_{h^{-1}} = \emptyset$

No solution

<u>Method 2</u>

From GC

When $h(x) = x$

$|\ln(2x)| + 1 = x$

From GC $x = 2.678 > 0.5$ (no solution)

<u>Method 3</u>

From GC

When $h(x) = x$

$-|\ln(2x)| + 1 = x$

From GC $x = 0.685 > 0.5$ (no solution)

3

(a) **(i)** $y = 1 + \sqrt{x} \Rightarrow x = (y-1)^2$

$f^{-1}(x) = (x-1)^2$

Domain of f^{-1} = Range of $f = [1, \infty)$

(ii) $\mathrm{ff}(x) = x$

$\mathrm{f}(1+\sqrt{x}) = x$

$1 + \sqrt{1+\sqrt{x}} = x$ …(1)

$\sqrt{1+\sqrt{x}} = x - 1$

$1 + \sqrt{x} = x^2 - 2x + 1$

$\sqrt{x} = x^2 - 2x$

$x = x^4 - 4x^3 + 4x^2$

$x^4 - 4x^3 + 4x^2 - x = 0$

$x(x^3 - 4x^2 + 4 - 1) = 0$

Note that $\mathrm{ff}(0) = \mathrm{f}(1) = 2$, thus $x = 0$ is not a solution $\mathrm{ff}(x) = x$

Thus, $x^3 - 4x^2 + 4 - 1 = 0$

From GC, $x = 1$, 0.382(3sf) or 2.62 (3sf)

It can be observed from (1) that $x > 2$, so $x = 2.62$.

Explanation:

Since f^{-1} exists,

$\mathrm{ff}(x) = x \Rightarrow \mathrm{f}^{-1}\mathrm{f}(\mathrm{f}(x)) = \mathrm{f}^{-1}(x)$

$\Rightarrow \mathrm{f}(x) = \mathrm{f}^{-1}(x)$ since $\mathrm{f}^{-1}\mathrm{f}(a) = a$

Thus this value of x satisfies $\mathrm{f}(x) = \mathrm{f}^{-1}(x)$.

(b) $$g(n) = \begin{cases} 1 & \text{for n =0,} \\ 2 + g\left(\frac{1}{2}n\right) & \text{for n = even,} \\ 1 + \mathrm{g}(\mathrm{n}-1) & \text{for n= odd} \end{cases}$$

(i) $g(4) = 2 + g(2) = 2 + 2 + g(1) = 4 + 1 + g(0) = 6$

$g(7) = 1 + g(6) = 1 + 2 + g(3) = 3 + 1 + g(2) = 4 + 4 = 8$

$g(12) = 2 + g(6) = 2 + 7 = 9$

(ii) $g(8) = 2 + g(4) = 2 + 6 = 8 = g(7)$ and $8 \neq 7$

∴ g is not $1-1$. Thus g does not have an inverse.

Observe from (i) that the values of g is integral and seems to "increase" when n increases from 7 and 12. But g only increases from 8 to 9. So we suspect there should exist repeated numbers.

4

$\mathrm{f} : x \mapsto \frac{x+a}{x+b}$, for $x \in \mathbb{R}, x \neq -b, a \neq -1$,

$\mathrm{g} : x \mapsto x$, for $x \in \mathbb{R}$.

Given that ff = g,

$\frac{\frac{x+a}{x+b}+a}{\frac{x+a}{x+b}+b} = x$

$\frac{x + a + ax + ab}{x + a + bx + b^2} = x$

$\frac{x(1+a) + a(1+b)}{x(1+b) + a + b^2} = x$

$x(1+a) + a(1+b) = x^2(1+b) + x(a+b^2)$

Comparing coefficients of x^2: $b = -1$

$\mathrm{ff}(x) = \mathrm{x}$

$\mathrm{f}^{-1}(x) = \mathrm{f}(x) = \frac{x+a}{x-1}$

5

(a) Let $y = \frac{3x+18}{2x-3}$,

$y(2x-3) = 3x + 18$

$2xy - 3y = 3x + 18$

$2xy - 3x = 3y + 18$

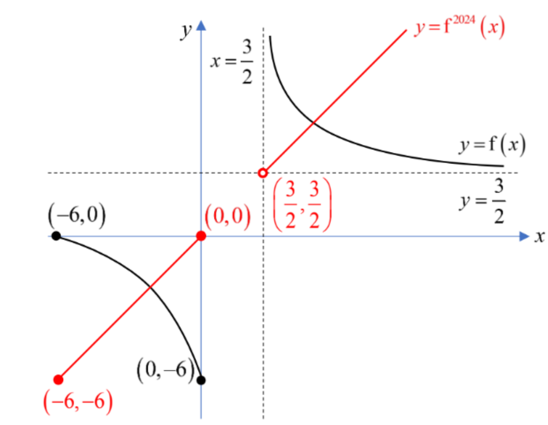
$x = \frac{3x+18}{2x-3}$

$\mathrm{f}^{-1}(x) = \frac{3x+18}{2x-3}$

$D_{f^{-1}} = R_f = [-6,0] \cup \left(\frac{3}{2}, \infty\right)$

(b) $[-6,0] \cup \left(\frac{3}{2}, \infty\right)$ OR $\left\{x \in \mathbb{R}: \ -6 \leq \mathrm{x} \leq 0, \mathrm{x} > \frac{3}{2}\right\}$

(c)



(c)

(d) $g^{23}(x) = e^{23}x$

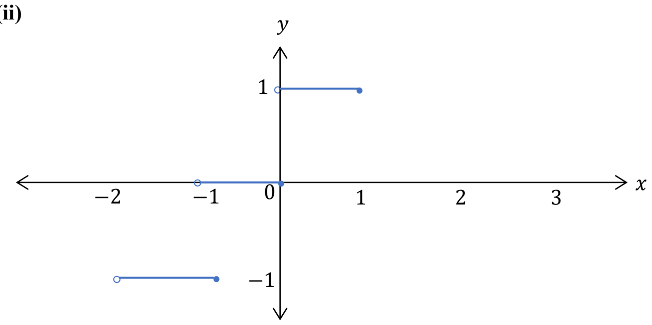
$g^{23}f(x) = e^{23}f(x)$

$D_{g^{23}f} = [-6e^{23}, 0] \cup \left(\frac{3}{2}e^{23}, \infty\right)$

6

(i) $f(-1.4) = -1$

(ii)



(iii) Method 1

No, because the horizontal line $y = 1$ (for example) cuts the graph more than once from $(0,1]$. So f is not $1-1$ so f^{-1} does not exist.

Method 2

No, because for example, $f(1.1) = f(1.2) = 0$. So f is not 1-1 so f^{-1} does not exist.

(iv) $R_f = \{-1,0,1\}$

(v) $g^2(x) = \frac{a\left(\frac{ax-3}{x-a}\right)-3}{\frac{ax-3}{x-a}-a}$

$= \frac{a^2x-3a-3x+3a}{ax-3-ax+a^2}$

$= x$

Then

$g^3(x) = g^2(g(x))$

$= \frac{ax-3}{x-a}$

Observe that even compositions give x, odd compositions give $g(x)$.

So, $g^{2019}(x) = \frac{ax-3}{x-a} \Rightarrow g^{2019}(5) = \frac{5a-3}{5-a}$.

(vi) $g(x) = \frac{3x-3}{x-3}$

$D_f = (-2,2] \xrightarrow{f} \{-1,0,1\} \xrightarrow{g} \left\{\frac{3}{2}, 1, 0\right\}$

7

(a) $\frac{d}{dx}(2x^2 - \lambda x - 3) = 0$

$4x - \lambda = 0 \qquad \Rightarrow x = \frac{\lambda}{4}$

OR

$2x^2 - \lambda x - 3 = 2\left(x^2 - \frac{\lambda}{2}x\right) - 3$

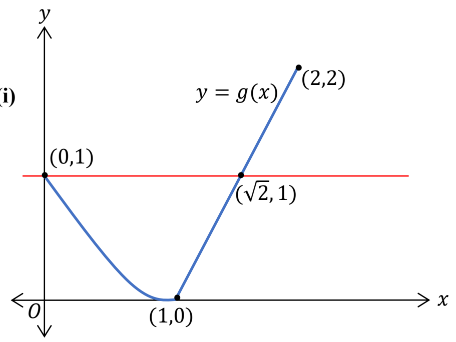
$= 2\left(x^2 + 2\left(-\frac{\lambda}{4}\right)x + \left(-\frac{\lambda}{4}\right)^2\right) - 3 - 2\left(-\frac{\lambda}{4}\right)^2$

$= 2\left(x - \frac{\lambda}{4}\right)^2 - -\frac{\lambda^2 + 24}{8}$

For f^{-1} to exist, the turning point of $y = 2x^2 - \lambda x - 3$ **cannot** lie in the interval $\frac{7}{4} < x < 5$. Hence,

$\frac{\lambda}{4} \leq \frac{7}{4}$ or $\frac{\lambda}{4} \geq 5 \Rightarrow \lambda \leq 7$ or $\lambda \geq 20$. Set of values is $(-\infty, 7] \cup [20, \infty)$ *or* $\{\lambda \in \mathbb{R}: \lambda \leq 7 \text{ or } \lambda \geq 20\}$.

(b) **(i)**



(ii) From the sketch in **(b)(i)**,
Point of intersection between $y = g(x)$ and $y = 1$ occurs at the points $(0, 1)$ and where
$2\log_2 x = 1 \Rightarrow \log_2 x = \frac{1}{2}$
$\Rightarrow x = 2^{\frac{1}{2}} = \sqrt{2}$
$1 < g(x) \le 2 \Rightarrow \sqrt{2} < x \le 2$

(iii) For $0 \le x \le 1, \ \ 0 \le g(x) \le 1$,
Hence $g^2(x) = g((x-1)^2) = ((x-1)^2 - 1)^2 = (x^2 - 2x)^2$.
Considering part **(b)(ii)**,
For $1 < x \le \sqrt{2}, 0 \le g(x) \le 1$,
hence $g^2(x) = g(2\log_2 x) = (2\log_2 x - 1)^2$
For $\sqrt{2} < x \le 2, \ 1 \le g(x) \le 2$,
Hence $g^2(x) = g(2\log_2 x) = 2\log_2(2\log_2 x)$

$$g^2(x) = \begin{cases} (x^2 - 2x)^2 & \text{for } 0 \le x \le 1, \\ (2\log_2 x - 1)^2 & \text{for } 1 < x \le \sqrt{2}, \\ 2\log_2(2\log_2 x) & \text{for } \sqrt{2} < x \le 2. \end{cases}$$

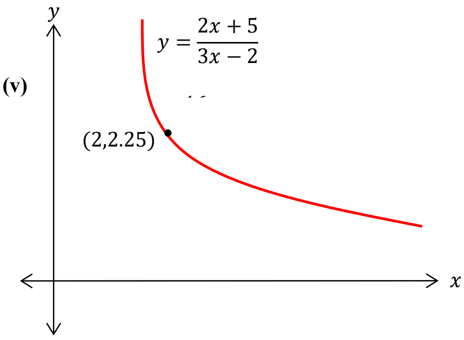
8 **(i)** Let $y = \frac{ax+b}{cx-a}, x \in \mathbb{R}, x \neq \frac{a}{c}$
$y(cx - a) = ax + b$
$x(cy - a) = ay + b$
$x = \frac{ay+b}{cy-a}$
$\therefore f^{-1}(x) = \frac{ax+b}{cx-a}, x \in \mathbb{R}, x \neq \frac{a}{c}$
$f = f^{-1}$ for $x \in \mathbb{R}, x \neq \frac{a}{c}$, hence f is self-inverse. (shown)

(ii) $f(x) = f^{-1}(x)$
Composing function f on both sides,
$ff(x) = ff^{-1}(x)$
$f^2(x) = x$
$D_{f^2} = \mathbb{R}\backslash\left\{\frac{a}{c}\right\}, \quad R_{f^2} = \mathbb{R}\backslash\left\{\frac{a}{c}\right\}$
or write as $R_{f^2} = \left(-\infty, \frac{a}{c}\right) \cup \left(\frac{a}{c}, \infty\right)$

(iii) $f^{-1}(x) = x$
$$\frac{ax+b}{cx-a} = x$$
$ax + b = cx^2 - ax$
$cx^2 - 2ax - b = 0$
$$x^2 - \frac{2a}{c}x - \frac{b}{c} = 0$$
$$\left(x - \frac{a}{c}\right)^2 = \frac{b}{c} + \left(\frac{a}{c}\right)^2$$
$$x - \frac{a}{c} = \pm\sqrt{\frac{bc + a^2}{c^2}} \Rightarrow x = \frac{a}{c} \pm \sqrt{\frac{bc + a^2}{c^2}} = \frac{a \pm \sqrt{bc + a^2}}{c},$$

(iv) Now. $a = 2$, $b = 5$ and $c = 3$
$f(x) = \frac{2x+5}{3x-2}, x \in \mathbb{R}, x \neq \frac{2}{3}$ and $g(x) = e^x + 2, \ x \in \mathbb{R}$.
For fg to exist, need $R_g \subseteq D_f$ to hold.
$R_g = (2, \infty) \subseteq \mathbb{R}\backslash\left\{\frac{2}{3}\right\} = D_f$, fg exists.

(v)



$D_g = \mathbb{R} \xrightarrow[g]{} R_g = (2, \infty) \xrightarrow[f]{} R_{fg} = \left(\frac{2}{3}, \frac{9}{4}\right)$