5 MUST KNOW QUESTIONS TO ACE GRAPH FUNCTIONS

Ellipse / Hyperbola / Rational Functions

1	**[YIJC/2021/Promo/Q10]** The curves C and D have equations $y=\frac{5}{3}-x-\frac{4}{x-3}$ and $\frac{(x-1)^2}{6^2}+\frac{y^2}{k^2}=1$ respectively, where k is a positive constant. **(i)** Using an algebraic method, find the exact range of values of y that C can take. **[4]** **(ii)** On the same axes, sketch **(a)** the graph of C, stating the equations of any asymptotes and the coordinates of the turning points, **[2]** **(b)** the graph of D for the case where $k = 2$, stating the coordinates of the centre, the turning points and the points of intersection with the x-axis. **[2]** **(iii)** State the exact range of values of k such that C and D intersect at more than one point. **[1]** **(iv)** State the range of values of m such that the line with equation $y+\frac{4}{3}=m(x-3)$ does not intersect C. **[4]**
2	**[VJC/2019/Promo/Q4]** The curve C has equation $\frac{(y-2)^2}{9}-\frac{(x-3)^2}{4}=1$. **(i)** Sketch C, giving the equations of its asymptotes and the coordinates of any turning points. **[4]** The curve D has equation $12y^2-48\text{y}+48+ax^2-6ax-3a=0$, where a is a positive constant. **(ii)** Find the set of values of a for which C and D do not intersect. **[4]**
3	**[ACJC/2022/Promo/Q3]** The curve C with equation $y=\frac{ax^2+bx+c}{3x+1}$ passes through the point with coordinates $(-1,-4)$ and has a turning point at $(-3,-2)$. **(i)** Find the values of a, b and c. **[3]** **(ii)** Hence sketch the graph of curve C, stating the equations of any asymptotes and the coordinates of any points where the curve crosses the axes **[3]**

4	**[NYJC/2023/Prelims/P1/Q3]** The curve C has equation $y = \frac{-x^2+ax+b}{2x+4}$,where a and b are constants. It is given that C has a turning point at the origin. **(a)** State the value of b and show that $a = 0$. **[3]** **(b)** By sketching the graph of C and adding a suitable curve onto the same diagram, find the two possible values of d such that the equation $(x-d)^2 + \frac{1}{4}\left(\frac{-x^2}{2x+4} - 2\right)^2 = 1$ has equal roots. **[3]**
5	**[RI/2022/Promo/Q7b]** A curve C has equation $y = \frac{2x^2+x+a}{x}$, **(i)** For a >1, sketch C, labelling clearly the equations of the asymptotes and the coordinates of the stationary points in terms of a where appropriate. **[4]** **(ii)** Given that the solution set of the inequality $\frac{2x^2+x+a}{x} \leq 0\ is\ (-\infty, 0)$, deduce the set of values of a. **[2]**

5 MUST KNOW QUESTIONS TO ACE GRAPH TRANSFORMATIONS

1 **[JPJC/2023/Prelim/P2/Q3]**

(a) Describe a sequence of two transformations which maps the graph of $y = e^x$ onto the graph of $y = -e^x + 2$. **[2]**

(b) The line with equation $y = x + 6$ undergoes the following transformations, in succession:

A: Scaling by a scale factor of $\frac{1}{3}$ parallel to the x-axis,

B: Translation of 2 units in the positive x-direction,

C: Scaling by a scale factor of $\frac{1}{3}$ parallel to the y-axis.

Find the equation of the resulting line. **[3]**

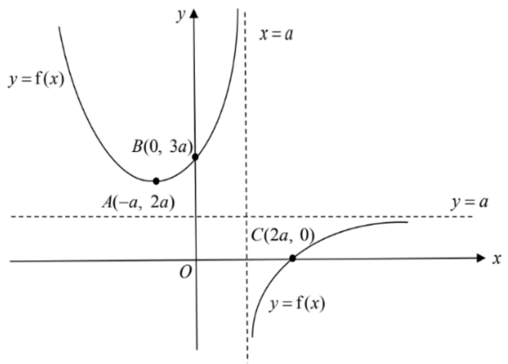
(c) The curve $y = f(x)$ cuts the axes at $(a, 0)$ and $(0, b)$. $x = c$ and $y = d$ are asymptotes of $f(x)$. It is given that $f^{-1}(x)$ exists. State the equations of the asymptotes and the coordinates of the axial intercepts of the following curves.

(i) $y = \frac{1}{f(x)}$, **[2]**

(ii) $y = f^{-1}(x)$, **[2]**

2 **[RVHS/2019/Prelims/P2/Q3]**

(a) The diagram below shows the graph of $y = f(x)$. The graph has a minimum point at $A(-a, 2a)$ where $a > 0$. It passes through the points $B(0, 3a)$ and $C(2a, 0)$. The equations of the asymptotes are $x = a$ and $y = a$. Each of the gradients of the tangents at points B and C is a.



Sketch on separate clearly labelled diagrams, the graphs of

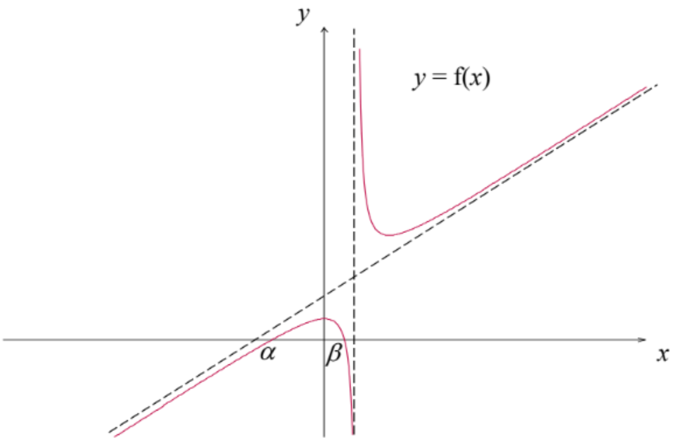
(i) $y = f(2x + a)$, **[2]**

(ii) $y = f'(x)$, **[2]**

(iii) $y = \frac{1}{f(x)}$. **[3]**

Label in each case, the coordinates of points corresponding to the points A, B and C (if any) of $y = f(x)$ and the equation of asymptote(s).

(b) Describe a sequence of transformations which transforms the graph of $2y^2 - x^2 = 1$ onto the graph of $y^2 - (x - 1)^2 = 1$. **[2]**

3	**[ASrJC/2021/Prelims/P1/Q7]**
	(i) The curve C_1 with equation $y = \frac{(x+2)^2}{x+1}$ is transformed onto the curve C_2 with equation $y = f(x)$. The curve C_1 has a minimum turning point $(0, 4)$ which corresponds to the point with coordinates (a, b) on the curve C_2, where $a, b > 0$. Given that $f(x)$ has the form $\frac{p^2x^2}{px-1} + q$. where p, q, are positive constants, express p and q in terms of a and b. **[4]**
	(ii) The curve of $y = f(x)$ has a maximum point and a minimum point at $(0, q)$ and (a, b) respectively, and intersects the x-axis at α and β, as shown in the diagram below. The equation of the vertical asymptote is $x = \frac{1}{p}$
	
	Sketch the curve $y = \frac{1}{f(x)}$. Your diagram should indicate clearly, in terms of a, b, α and β, the equations of any asymptote(s) as well as the coordinates of turning points and axial intercepts. **[3]**
4	**[N2016/P1/Q3]**
	The curve $y = x^4$ is transformed onto the curve with equation $y = f(x)$. The turning point on $y = x^4$ corresponds to the point with coordinates (a, b) on $y = f(x)$. The curve $y = f(x)$ also passes through the point with coordinates $(0, c)$. Given that $f(x)$ has the form $k(x-1)^4 + m$ and that a, b and c are positive constants with $c > b$, express k, I and m in terms of a, b and c. **[2]**
	By sketching the curve $y = f(x)$, or otherwise, sketch the curve $y = \frac{1}{f(x)}$. State, in terms of a, b and c, the coordinates of any points where $y = \frac{1}{f(x)}$ crosses the axes and of any turning points. **[4]**
5	**[N2021/P1/Q6]**
	A curve C has equation $y = \frac{1}{\sqrt{4ax-x^2}}$, where $a > 0$.
	(a) Sketch C and give the equations of any asymptotes, in terms of a where appropriate. **[4]**
	(b) Find the smallest possible value of y in terms of a. **[1]**
	(c) Describe the transformation that maps the graph of C onto the graph of $y = \frac{1}{\sqrt{4ax-x^2}}$ **[3]**

Solutions

1

(i)

$y = \frac{5}{3} - x - \frac{4}{x-3}$

$(x-3)y = (x-3)\left(\frac{5}{3} - x\right) - 4$

$xy - 3y = \frac{5}{3}x - x^2 - 5 + 3x - 4$

$3xy - 9y = -3x^2 + 14x - 27$

$3x^2 + (3y-14)x + (27-9y) = 0$

The equation has real roots $\Longrightarrow 3b^2 - 4ac \geq 0$

$(3y-14)^2 - 4(3)(27-9y) \geq 0$

$9y^2 - 84y + 196 - 324 + 108y \geq 0$

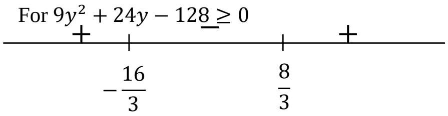
$9y^2 + 24y - 128 \geq 0$

Consider $9y^2 + 24y - 128 \geq 0$

$y = \dfrac{-24 \pm \sqrt{24^2 - 4(9)(-128)}}{18}$

$= \frac{8}{3}$ or $-\frac{16}{3}$

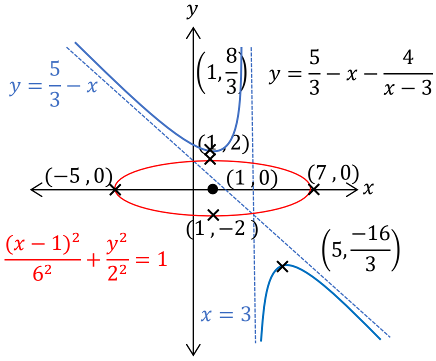
For $9y^2 + 24y - 128 \geq 0$



The range of values that C can take is

$y \leq \frac{16}{3}$ or $y \geq \frac{8}{3}$

(ii)



(iii) For C and D to intersect at more than one point, $k > \frac{8}{3}$

(iv) The line $y + \frac{4}{3} = m(x-3)$ has gradient m and passes through the point $\left(3, -\frac{4}{3}\right)$ which is the point of intersection of the vertical and oblique asymptotes. From the graph, the line does not intersect C when $m \geq -1$.

2

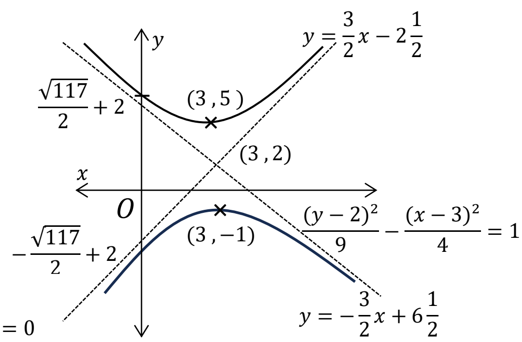
(i) Asymptotes:

$\frac{(y-2)^2}{9} = \frac{(x-3)^2}{4}$

$\frac{(y-2)}{3} = \pm\frac{(x-3)}{2}$

$y = \pm\frac{3(x-3)}{2} + 2$

$y = \frac{3}{2}x - \frac{5}{2}$ or $y = -\frac{3}{2}x + \frac{13}{2}$



(ii)

$12y^2 - 48y + 48 + ax^2 - 6ax - 3a = 0$

$12(y^2 - 4y) + a(x^2 - 6x) - 3a + 48 = 0$

$12(y^2 - 4y + 4) - 48 + a(x^2 - 6x + 9) - 9a - 3a + 48 = 0$

$12(y-2)^2 + a(x-3)^2 = 12a$

$\frac{(y-2)^2}{a} + \frac{(x-3)^2}{12} = 1$

which is an ellipse with centre (3,2) and vertices at $(3, 2+\sqrt{a})$ and $(3, 2-\sqrt{a})$

For curve C and D to not intersect, $\sqrt{a} < 3 \Rightarrow a < 9$

Since a is a positive constant, $\therefore \{a \in \mathbb{R}: 0 < a < 9\}$

3

(i) $y = \frac{ax^2+bx+c}{3x+1}$

Substitute $(-1,-4)$ and $(-3,-2)$ into equation,

$a - b + c = 8$ ---- (1)

$9a - 3b + c = 16$ ---- (2)

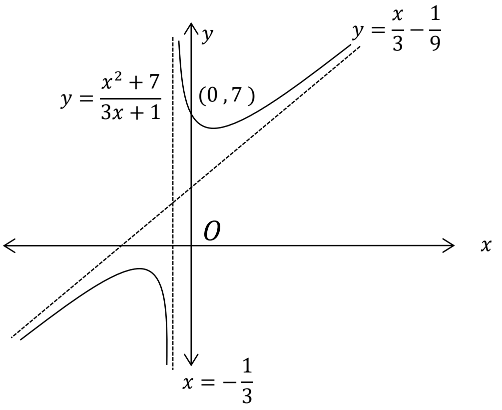
$$\frac{dy}{dx} = \frac{(2ax+b)(3x+1) - 3(ax^2+bx+c)}{3x+1}$$

Substitute $x = -3$ and $\frac{dy}{dx} = 0$

$0 = (-6a+b)(-8) - 3(9a - 3b + c)$

$21a + b - 3c = 0$ ---- (3)

From GC, $\mathrm{a} = 1, \mathrm{b} = 0$ and $\mathrm{c} = 7$



(ii) $y = \frac{x^2+7}{3x+1}$

$= \frac{x}{3} - \frac{1}{9} + \frac{64}{9(3x+1)}$

$$\begin{array}{r}
\frac{x}{3} - \frac{1}{9} \\
3x+1\,\overline{)\,x^2 + 0x + 7} \\
x^2 + \frac{x}{3} \\
\hline
-\frac{x}{3} + 7 \\
-\frac{x}{3} - \frac{1}{9} \\
\hline
7\frac{1}{9}
\end{array}$$

4

(a) **Method 1:**

Let $y = 0$

$-x^2 + ax + b = 0$

$x^2 - ax - b = 0$

$\left(x - \frac{a}{2}\right)^2 - b - \left(\frac{a}{2}\right)^2 = 0$

Since $(0, 0)$ is a turning point, $a = \mathrm{b} = 0$

Method 2:

Since $(0, 0)$ is a turning point,

$$0 = \frac{0+0+b}{4}$$

$b = 0$

$$y = \frac{-x^2 + ax + 0}{2x+4}$$

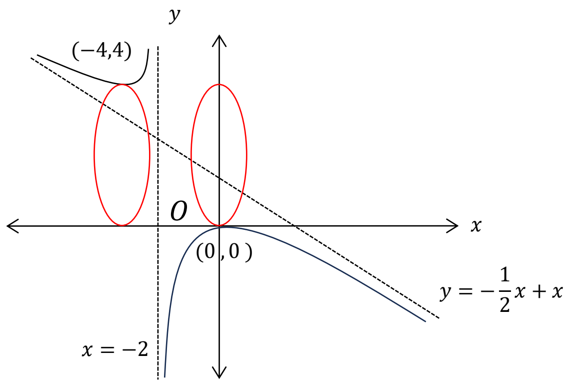
$= -\frac{1}{2}x + \frac{a+2}{2} + \frac{-2a-4}{2x+4}$

$$\begin{array}{r}
-\frac{1}{2}x + \frac{a+2}{2} \\
2x+4\,\overline{)\,-x^2 + ax} \\
-x^2 - 2x \\
\hline
(a+2)x \\
(a+2)x + 2a + 4
\end{array}$$

$$\frac{dy}{dx} = -\frac{1}{2} + \frac{4a+8}{(2x+4)^2}$$

Since $(0, 0)$ is a turning point, $0 = -\frac{1}{2} + \frac{4a+8}{16}$

$a = 0$



Since turning points are $(0, 0)$ and $(-4, 4)$ $d = -4$ or $d = 0$

(Note: Ellipse must meet C at exactly one point)

(b) $(x-d)^2 + \frac{1}{4}\left(\frac{-x^2}{2x+4} - 2\right)^2 = 1$

$(x-d)^2 + \left(\frac{y-2}{2}\right)^2 = 1$

5

(i) $y = \frac{2x^2+x+a}{x} = 2x + 1 + \frac{a}{x}$

$\Rightarrow$ Asymptotes are $y = 2x + 1$ and $x = 0$.

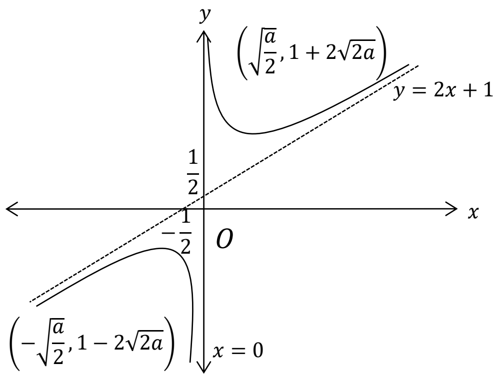
$$\frac{dy}{dx} = 2 - \frac{a}{x^2}$$

$\frac{dy}{dx} = 0 \Leftrightarrow 2 - \frac{a}{x^2} = 0 \quad \Leftrightarrow x^2 = \frac{a}{2}$

Since a > 1 > 0, $\frac{dy}{dx} = 0$ has 2 distinct solutions, hence the graph has 2 distinct stationary points.

When $x = \pm\sqrt{\frac{a}{2}}, y = 2\left(\pm\sqrt{\frac{a}{2}}\right) + 1 \pm \frac{a}{\sqrt{\frac{a}{2}}} = 1 \pm 2\sqrt{2a}$

Coordinates of stationary points are
$\left(\sqrt{\frac{a}{2}}, 1 + 2\sqrt{2a}\right)$ *and* $\left(-\sqrt{\frac{a}{2}}, 1 - 2\sqrt{2a}\right)$



(ii) For solution set of $\frac{2x^2+x+a}{x} \leq 0$ to be $(-\infty, 0)$,
$2x^2 + x + a \geq 0$ for all $x \in (-\infty, 0)$.

Require $2x^2 + x + a = 2\left(x + \frac{1}{4}\right)^2 + a - \frac{1}{8} \geq 0$
$\Rightarrow a \geq \frac{1}{8}$

Alternatively, using graph we require $1 - 2\sqrt{2a} \leq 0$
$\Rightarrow \sqrt{a} \geq \frac{1}{2\sqrt{2}} \Rightarrow a \geq \frac{1}{8}$
Set of values of $a = [\frac{1}{8}, \infty)$.

Solutions

1

(a) Method 1

$y = e^x \to y = -e^x \to y = -e^x + 2$

Reflect about the x-axis.

Translate by 2 units in the positive y-direction.

Method 2

$y = e^x \to y = e^x - 2 \to y = -(e^x - 2) = -e^x + 2$

Translate by 2 units in the positive y-direction.

Reflect about the x-axis.

(b) After A: $y = 3x + 6$

After B: $y = 3(x-2) + 6 = 3x$

After C: $y = \frac{1}{3}(3x) = x$

(ci) Asymptotes: $x = a$, $y = \frac{1}{d}$

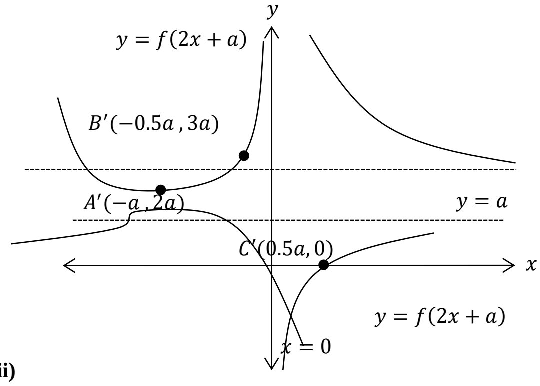
Axial intercepts: $(c, 0)$, $\left(0, \frac{1}{b}\right)$

(cii) Asymptotes: $x = d$, $y = c$

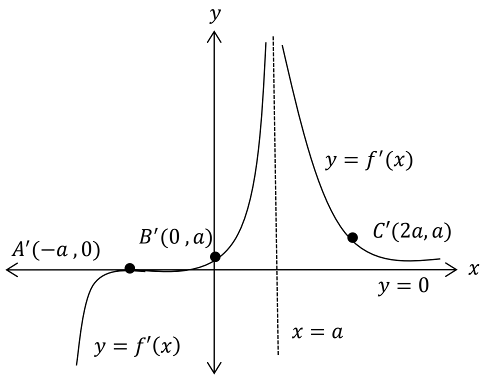
Axial intercepts: $(0, a)$, $(b, 0)$

2

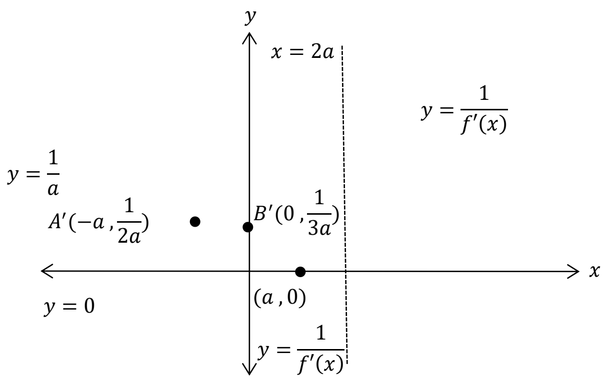
(ai) $f(x) \to f(x+a) \to f(2x+a)$



(aii)



(aiii)



(b) We first note the following transformation steps:

$2y^2 - x^2 = 1$

Step 1: Replace 'x' by '$x - 1$'

$2y^2 - x^2 = 1$

Step 2: Replace 'y' by '$\frac{y}{\sqrt{2}}$'

$$2\left(\frac{y}{\sqrt{2}}\right)^2 - (x-1)^2 = 1$$
$$y^2 - (x-1)^2 = 1$$

Thus, the sequence of transformations needed are as follow
1. A translation of 1 unit in the positive x axis direction;
2. A scaling parallel to the y axis of factor $\sqrt{2}$.

3

(i) $y = \frac{(x+2)^2}{x+1}$ —Replace x with $x-2$→ $y = \frac{[(x-2)+2]^2}{(x-2)+1} = \frac{x^2}{x-1}$ (Translation of 2 units in the positive x-direction)

—Replace x with px→ $y = \frac{(px)^2}{(px)-1} = \frac{p^2x^2}{px-1}$ (Scaling of scale factor $\frac{1}{p}$ along the x-axis)

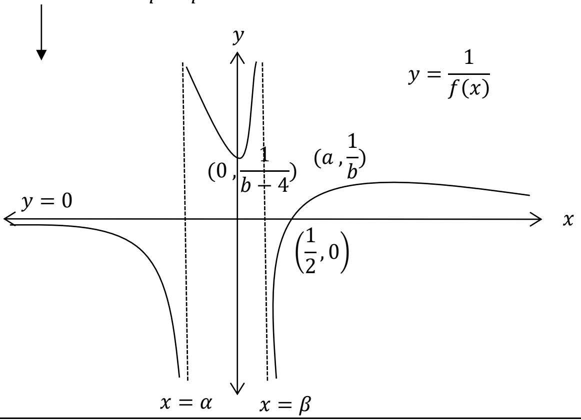
—Replace y with $y-q$→ $y = \frac{p^2x^2}{px-1} = \frac{p^2x^2}{px-1}$ (Translation of q units in the positive y-direction)

The minimum turning point on C_1 $(0,4)$ corresponding to (a,b) on C_2.

Hence, $a = \frac{1}{p}(0+2) \rightarrow p = \frac{2}{a}$

$b = 4 + q \rightarrow q = b - 4$

(ii)

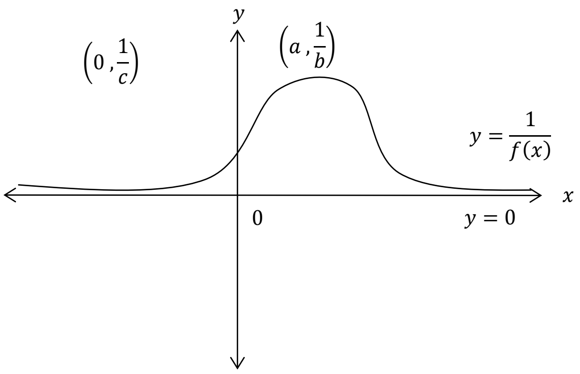


4

$f(x) = k(x-l)^4 + m$
$f'(x) = 4k(x-l)^3$
$f'(a) = 0 \Rightarrow 4k(a-l)^3 = 0$
$\Rightarrow l = a \text{ since } k \neq 0$

Thus $f(x) = k(x-a)^4 + m$
$f(a) = b \Rightarrow k(a-a)^4 + m = b$
$\Rightarrow m = b$

Thus $f(x) = k(x-a)^4 + b$
$f(0) = c \Rightarrow k(0-a)^4 + b = c$
$\Rightarrow k = \frac{c-b}{a^4}$



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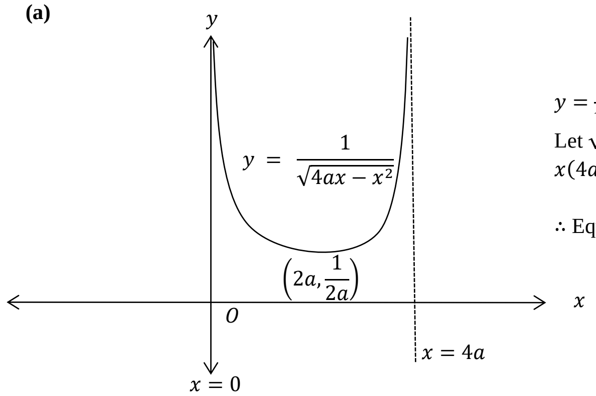
(a)



$y = \frac{1}{\sqrt{4ax-x^2}}, a > 0$

Let $\sqrt{4ax - x^2} = 0 \Rightarrow 4ax - x^2 = 0$

$x(4a - x) = 0$

$\therefore$ Equation of vertical asymptotes are $x = 0$ or $x = 4a$

(b) Let $y_1 = 4ax - x^2 = 4a^2 - (x - 2a)^2$

Max $y_1 = 4a^2$ when $x = 2a$

$\therefore$ when $x = 2a, min\ y = \frac{1}{\sqrt{4a^2}} = \frac{1}{2a} (since\ a > 0)$

(c) $y = \frac{1}{\sqrt{4ax-x^2}} = \frac{1}{\sqrt{4ax-(x-2a)^2}}$

By replacing x with $x + 2a$, we get $y = \frac{1}{\sqrt{4a^2-x^2}}$

$\therefore$ Translate the graph of C by $2a$ units in the negative x-direction